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Uniform Approximation by Linear Combinations of Bernstein-Type Polynomials

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Uniform approximation is considered by linear combinations due to May and Rathore of integral modifications of the Bernstein polynomial introduced by Durrmeyer. The order of uniform approximation is obtained in terms of higher-order modulus of continuity of the function being approximated.

1. Introduction

Durrmeyer [3] introduced the following integral modification of the Bernstein polynomial:

$$M_n(f(t), x) = (n+1) \sum_{v=0}^{n} P_{nv}(x) \int_0^1 f(t)\, P_{nv}(t)\, dt, \tag{1.1}$$

where

$$P_{nv}(x) = \binom{n}{v} x^v (1-x)^{n-v}, \qquad v = 0, 1, \ldots, n.$$

This operator is, of course, similar to the well-known Bernstein–Kantorovitch polynomial. Recently Derriennic [1] obtained the estimate

$$\|M_n(f) - f\| \leqslant 2\omega(f, n^{-1/2}), \qquad n = 3, 4, 5, \ldots, \tag{1.2}$$

where $f \in C(I)$, the space of functions continuous on $I = [0, 1]$, $\|\cdot\|$ denotes the uniform norm on I, and $\omega(f, \cdot)$ is the ordinary modulus of continuity of f on I.

May [4] and Rathore [5] have described a method for forming linear combinations of positive linear operators so as to improve the order of

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approximation. We apply this technique to (1.1) in order to improve estimate (1.2). The approximation process is described as follows:

$$L_n(f(t), k, x) = \sum_{j=0}^{k} c(j, k)\, M_{d_j n}(f(t), x), \tag{1.3}$$

where $d_0, d_1, \ldots, d_k$ are $k+1$ arbitrary, fixed, and distinct positive integers and

$$c(j, k) = \prod_{\substack{i=0 \\ i \neq j}}^{k} \frac{d_j}{d_j - d_i}, \qquad k \neq 0 \text{ and } c(0, 0) = 1. \tag{1.4}$$

In the next section we establish

$$\|L_n(f, k, \cdot) - f\| \leqslant M_k(n^{-(k+1)}\|f\| + \omega_{2k+2}(f, n^{-1/2})), \tag{1.5}$$

for all n sufficiently large, where $f \in C(I)$, M_k is a positive constant that depends on k but is independent of f and n, and $\omega_{2k+2}(f, \cdot)$ denotes the $2k+2$ order modulus of continuity of f on I [6, p. 102]. Estimate (1.5) improves (1.2) even for $k = 0$ when L_n reduces to M_n.

In a separate work [7] the author has established order of approximation by (1.3) in the space, $L_p(I)$ $(1 \leqslant p < \infty)$, of pth power Lebesgue integrable functions on I.

2. Order of Uniform Approximation

We require two lemmas, the first of which follows from calculations found in [1].

Lemma 2.1. *For $x \in I$, $j = 1, 2, \ldots$, $n = 1, 2, \ldots$, and $f \in C(I)$,*

$$(n+1) \sum_{v=0}^{n} P_{nv}(x) \int_0^1 P_{nv}(t)\, dt = 1, \tag{2.1}$$

$$M_n((t-x)^{2j}, x) \leqslant Cn^{-j}, \tag{2.2}$$

where the constant C depends on j but is independent of x and n, and

$$\|M_n(f)\| \leqslant \|f\|. \tag{2.3}$$

Lemma 2.2. *Let $e_v(x) = x^v$, $v = 0, 1, 2, \ldots$. For $x \in I$ and $v = 0, 1, 2, \ldots, 2k+2$,*

$$L_n(e_0, k, x) = 1 \tag{2.4}$$

and

$$\|L_n(e_v, k, \cdot) - e_v\| = O(n^{-(k+1)}), \qquad n \to \infty. \tag{2.5}$$

Proof. Result (2.4) follows from (1.3), (2.1), (1.4) and [4, p. 1228].

It follows from [1] that, for each $x \in I$ and each $v = 1, 2,..., M_n((t-x)^v, x)$ can be expressed as a rational function in n. The degree of the numerator is less than the degree of the denominator and the degree of both numerator and denominator depends on v. The denominator is independent of x and has distinct integer roots. The coefficients of the polynomial in n in the numerator are polynomials in x of degree at most v. Using partial fractions and (1.3) we obtain polynomials $a_s(x)$ of degree at most v and distinct integers α_s, $s = 1,..., g(v)$, such that for $x \in I$, $v = 1, 2,..., 2k+2$ and all n sufficiently large,

$$\begin{aligned} L_n((t-x)^v, k, x) &= \sum_{j=0}^{k} c(j,k) \sum_{s=1}^{g(v)} \frac{a_s(x)}{d_j n - \alpha_s} \\ &= \sum_{s=1}^{g(v)} a_s(x) \sum_{r=0}^{\infty} \frac{\alpha_s^r}{n^{r+1}} \sum_{j=0}^{k} c(j,k)\, d_j^{-(r+1)}. \end{aligned}$$

Using the above, (1.4) and [4, p. 1228] we obtain

$$L_n((t-x)^v, k, x) = \frac{1}{n^{k+1}} \left\{ \sum_{s=1}^{g(v)} a_s(x)\, \alpha_s^k \sum_{j=0}^{k} \frac{c(j,k)}{d_j^{k+1}} \left(1 - \frac{\alpha_s}{d_j n}\right)^{-1} \right\}.$$

Therefore, for $x \in I$, $v = 1, 2,..., 2k+2$, and all n sufficiently large,

$$|L_n((t-x)^v, k, x)| \leqslant C_k n^{-(k+1)}, \tag{2.6}$$

where C_k is a positive constant that depends on k but is independent of n. Finally, for $v = 1, 2,..., 2k+2$,

$$L_n(e_v, k, x) = x^v L_n(e_0, k, x) + \sum_{j=1}^{v} \binom{v}{j} x^{v-j} L_n((t-x)^j, k, x)$$

and (2.5) follows from (2.4) and (2.6).

THEOREM 2.3. *If $f \in C(I)$ then, for all n sufficiently large,*

$$\|L_n(f, k, \cdot) - f\| \leqslant M_k(n^{-(k+1)} \|f\| + \omega_{2k+2}(f, n^{-1/2})),$$

where M_k is a positive constant that depends on k but is independent of f and n.

Proof. Let $g \in C^{(2k+2)}(I)$, the space of functions with continuous derivatives or order $2k+2$ on I. For $x, t \in I$ construct [2, p. 25]

$$U(g, x, t) = \sum_{i=0}^{2k+2} A_i(g, x)\, e_i(t)$$

satisfying

$$\left.\frac{\partial^i U}{\partial t^i}\right|_{t=x} = g^i(x), \qquad i = 0, 1,..., 2k+2, \tag{2.7}$$

$$|A_i(g, x)| \leqslant \gamma_1(\|g\| + \|g^{(2k+2)}\|), \qquad i = 0, 1,..., 2k+2, \tag{2.8}$$

and

$$|g(t) - U(g, x, t)| \leqslant \gamma_2(\|g\| + \|g^{(2k+2)}\|)(t-x)^{2k+2}, \tag{2.9}$$

where γ_1 and γ_2 are absolute constants.

We have

$$\begin{aligned} |L_n(g(t), k, x) - g(x)| \leqslant {} & |L_n(g(t) - U(g, x, t), k, x)| \\ & + |L_n(U(g, x, t), k, x) - U(g, x, x)| \\ & + |U(g, x, x) - g(x)|. \end{aligned} \tag{2.10}$$

The last term on the right side of (2.10) is zero by (2.7), while (1.4), (2.2) and (2.9) imply that the first term on the right side of (2.10) is dominated by

$$\gamma_2^1(\|g\| + \|g^{(2k+2)}\|)\, n^{-(k+1)},$$

where γ_2^1 is a constant that depends on k but is independent of x and n. Using (2.5) and (2.8), the middle term on the right side of (2.10) is dominated by

$$\gamma_1^1(\|g\| + \|g^{(2k+2)}\|)\, n^{-(k+1)},$$

for all n sufficiently large, where γ_1^1 is a constant that depends on k but is independent of x and n.

Therefore, for $x \in I$ and all n sufficiently large,

$$|L_n(g(t), k, x) - g(x)| \leqslant (\text{const.})(\|g\| + \|g^{(2k+2)}\|)\, n^{-(k+1)},$$

where the constant on the right side depends on k but is independent of n. Using (1.3), (1.4) and (2.3) it is easy to see that $\{L_n\}$ is a uniformly bounded sequence of linear operators on $C(I)$. The estimate of Theorem 2.3 now follows by interpolation [2, p. 27].

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